Course Material on Complex analysis

Cauchy Riemann Equations

The Cauchy Riemann equations can be written in several ways. Given a complex function $f(x, y) = u(x, y) + iv(x, y)$, to be complex differentiable, we must have the limit of the drivatives to exist and unique at a given point z_0 in the complex plane. This means:

$$\lim_{z \to z_0} \frac{f(x, y) - f(x_0, y_0)}{z - z_0} = \text{unique and } < \infty$$
$$\lim_{\triangle x \to 0} \lim_{\triangle y \to 0} \frac{f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)}{\triangle x + i\triangle y}, \quad z = z_0 + \triangle x + i\triangle y \tag{1}$$

Since the limit should be independent of order we can take the following : $\triangle y \to 0$ first and then $\triangle x \to 0$ and vice versa and the answer should be independent of order,hence equating both we obtain at point z_0 :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2}$$

1. Use the above and the fact that $x = \frac{z+z^*}{2}$ and $y = \frac{z-z^*}{2i}$. i.e. a linear transformation to show for a function satisfying Cauchy Riemann equations we must have:

$$\frac{\partial f}{\partial z^*} = 0 \tag{3}$$

2. More over show that for a pair of functions which satisfy above, we must have:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{4}$$

3. Derive the Cauchy Reimann equations in polar coordintes (r, θ), where we have: $r^2 = x^2 + y^2$ and $\tan\theta = y/x$.
4. Find the imaginary partof the function :

$$\ln\left[\frac{1 + z^n}{1 - z^n}\right] \quad n \in N \tag{5}$$

5. Given the real part of a function $f(z)$ satisfying Cauchy Riemann equation (called Holomorphic functions), $u(x, y) = e^{-x}(x \sin y - y \cos y)$, find the imagibary part $v(x, y)$ by directly using Cauchy Reimann equations and again by writing the $u(x, y) = \Re[f(z)] = (f(z) + f(z^*)/2$ and hence deducing the form of $v(x, y)$

Complex Integration

Complex integration is understood as a Riemann integral (splitting the path up into segments $z_{k+1} - z_k$), i.e. as an average value of the function over a given curve/path :

$$\lim_{n \to \infty} \sum_{k=1}^{n-1} f(z_k)[z_{k+1} - z_k] = \int_{z_1=z_a}^{z_n=z_b} f(z)dz \tag{6}$$

6. Show that this can be written as follows :

$$L \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} f(z_k) e^{i\theta_k} \tag{7}$$

Where L is the length of the curve over which the complex integration is performed and $e^{i\theta_k} = [z_{k+1} - z_k]/|[z_{k+1} - z_k]|$

7. Using the definiton of the Riemann integral, prove Darboux inquality, which states the modulus of the value of complex integration over a curve is bounded by the product of the upper bound of the function over the curve M, and the length of the curve L.

8. Prove Cauchy Integral theorem for a simply connected region, which states that : given an analytic function $f(z)$ in a closed region R and $f'(z)$ be continuous in R. Let C be a simple closed contour in R, then

$$\oint_C f(z)dz = 0 \tag{8}$$

A) Using Greens theorem B) By first proving existence of Taylor series from analyticity and hence using it .

9. Generalize the above for non-simple contours and multiply connected regions.

10. Show that the line integral of a function $f(z)$ between two points lying inside a region R where $f(z)$ is analytic, is independent of the path.

11. Prove Cauchy's Integral Formula, which states that: Given a function $f(z)$ which is analytic inside a simple contour C, the value of the function $f(z)$ at any point z_0 in the interior of C is given by the following contour integral :

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \tag{9}$$

Hence show that :

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{10}$$

12 TAYLOR SERIES : Let a function $f(z)$ be analytic within a circle C of radius R and with center at $z = z_0$. Then it can be expanded as a series at any point z within C as :

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + \ldots \tag{11}$$

This is done by performing a series expansion of the denominator of Cauchy's Integral formula. Also prove the convergence of the series directly.

Non-Analytic functions

Since a closed circular contour C is a domain over which a given complex function has to be periodic, we must have the Fourier expansion of the function over C as:

$$f(z_0 + r^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta} \tag{12}$$

where z_0 is the center of the contour C and r is the radius. Here $a_n r^n$ are understood as the Fourier coefficients. If this is true for *any* $r > \epsilon$ around the point z_0, such that the Fourier series converges (which is an obvious requirement for the Fourier series), then we have $re^{i\theta} = z - z_0$ and the above becomes:

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{13}$$

It is clear from this generic expansion that there might be a singularity at $z = z_0$ since for negative n there is a divergence. This leads us to look at Non-analytic functions.

Non-Analytic functions are functions which are either singular at some point z_0 on the complex plane or is not single valued along a curve. These are classified as follows:

a) Isolated singularity : In some region R, if the function $f(z)$ is non-analytic only at a point i.e. $|f(z| < \infty$ at points $z \neq z_0$ and single valued, and singular only as $\lim_{z\to z_0} f(z) \to \infty$, then the singularity is called isolated. This is moreover classified into two parts
i) Poles : If the singularity is of finite order i.e. the behaviour of the function around the singular point is of the following form :

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n \tag{14}$$

This is called a pole of order N. More exactly there exists a smallest N such that :

$$\lim_{z\to z_0} (z - z_0)^N f(z) < \infty \text{ and } \neq 0 \tag{15}$$

This means the above limit is finite and non-zero only for a particular N which the most divergent term. The examples of these kind are as follows:

$$\frac{1}{(z-1)^3} \qquad \text{Pole of order 3} \tag{16}$$

The $N = 1$ case is called the simple pole and is important due to several reasons as we will see latter.
ii)If there exists no such N i.e. there is no lower limit for the series above to truncate then it is called an essential singularity at $z = z_0$,

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{17}$$

We can look at the example :

$$\sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots \text{series does not truncate} \tag{18}$$

b) We might also have a situation where the function is not continuous along a curve on the complex plane , i.e. the limiting value of the function ($\pm\epsilon$ away from the curve) as we approach the curve from both sides is not the same. This is called a Branch Cut singularity. The nature of the Branch cut depends on the pramaterization of the complex plane which is chosen. Examples of this are as follows:

$$\begin{aligned}
&i)\ln(z) \quad \text{for the parametrization } -\pi \leq \theta \leq \pi \\
&\text{ check that the branch cut is along the negative real axis} \\
&ii)\ln(z) \quad \text{for the parametrization } 0 \leq \theta \leq 2\pi \\
&\text{ check that the branch cut is along the positive real axis} \\
&iii)\sqrt{z^2-1} \quad \text{for different parametrizations}
\end{aligned} \tag{19}$$

13 LAURENT SERIES : Let a function have an isolated singularity at $z = z_0$, enclosed in a circle $C_1 : |z - z_0| = R_1$. More over C_2 be a circle with center z_0 of radius $R_2 > R_1$, such that $f(z)$ is analytic on C_1, C_2 and within the annular region between C_1 and C_2. Then for any z in the annular domain, we have the series expansion of $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} \tag{20}$$

Where :

$$\begin{aligned}
a_n &= \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)dz}{(z - z_0)^{n+1}} \\
a_{-n} &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)dz}{(z - z_0)^{-n+1}}
\end{aligned} \tag{21}$$

14 Prove that the Laurent series is unique in a given annular domain.

15 Find the Laurent series of the follwing functions in the region mentioned and state the kind of singularity:

$$\begin{aligned}
&i)\frac{e^{2z}}{(z-1)^3} && \text{about z=1} \\
&ii)(z-3)\sin\frac{1}{z+2} && \text{about z=-2} \\
&iii)\frac{z - \sin z}{z^3} && \text{about z=0} \\
&iv)\frac{z}{(z+1)(z+2)} && \text{about z=-2} \\
&v)\frac{1}{z^2(z-3)^2} && \text{about z=3}
\end{aligned} \tag{22}$$

Let us do the last problem, which means we have to find a power series in the parmeter $z-3$. To do this we simply use binomial expansion:

$$\begin{aligned}
&\frac{1}{z^2(z-3)^2} \\
&= \frac{1}{(3+(z-3))^2(z-3)^2} \\
&= \frac{1}{(z-3)^2 9(1+\frac{(z-3)}{3})^2} \\
&= \frac{1}{9(z-3)^2}[1+(-2)\frac{(z-3)}{3} + \frac{(-2)(-2-1)}{2!}\frac{(z-3)^2}{3^2} + \ldots] \\
&= \frac{1}{9(z-3)^2} - \frac{2}{27}\frac{1}{(z-3)} + \frac{1}{27} + \ldots
\end{aligned} \tag{23}$$

16 Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for the following regions:

$$\begin{aligned}
&a) 1 < |z| < 3 \\
&b) |z| > 3 \\
&c) 0 < |z+1| < 2 \\
&d) |z| < 1
\end{aligned} \tag{24}$$

Let us do the first example. This is a annular region bounded by the circles centered around origin of radius 1 and 3 respectively. Thus we have:

$$\begin{aligned}
&\frac{1}{(z+1)(z+3)} \\
&= \frac{1}{2}\frac{2}{(z+1)(z+3)} \\
&= \frac{1}{2}\frac{(z+3)-(z+1)}{(z+1)(z+3)} \\
&= \frac{1}{2(z+1)} - \frac{1}{2(z+3)}
\end{aligned} \tag{25}$$

Now since $|z| > 1$ we expand the first term in a geometric series as :

$$\begin{aligned}
&\frac{1}{1+z} \\
&= \frac{1}{z(1+\frac{1}{z})} \\
&= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \ldots
\end{aligned} \tag{26}$$

Agian for the next term we have $|z| > 3$. Hence we expand it as follows:

$$\begin{aligned} &\frac{1}{z+3} \\ &= \frac{1}{3(1+\frac{z}{3})} \\ &= \frac{1}{3}(1 - \frac{z}{3} + \frac{z^2}{9} - \dots) \end{aligned} \tag{27}$$

Plugging this two expansions in we have:

$$\dots + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \dots \tag{28}$$

Please note that although there exists all the power of z here including the negative ones, this does not mean that there is an essential singularity at $z = 0$. This is because the expansion is only valid in the region $1 < |z| < 3$ and not outside. Hence the expansion diverges for all points $|z| \leq 1$, and hence this Laurent series can not be continued to the point $z = 0$.

Cauchy Residue Theorem

Let $f(z)$ be analytic on and within a closed contour C taken anticlockwise except for a finite set of isolated points $z_1, z_2, \dots, z_n$ at which it is singular with residues : $a_{-1}^{(1)}, \dots, a_{-1}^{(n)}$. Then :

$$\oint_C f(z)dz = 2\pi i \sum_{i=1}^{n} a_{-1}^{(i)} \tag{29}$$

where $a_{-1}^{(i)}$ are the Laurent Series coefficients corresponding to the simple poles $(z - z_i)^{-1}$. To prove this we choose a multiply connected contour Γ which is composed of C in the anticlockwise direction and $-C_i$ (C_i are defined as anticlockwise around the points z_i with radius ϵ_i. The minus sign is taken to make it clockwise). Thus this encloses a region where $f(z)$ is analytic, (note that the region is always to the left of the contour Γ and hence of C and $-C_i$). Thus using Cauchy's Theorem for multiply connected regions, we have:

$$\oint_\Gamma f(z)dz = 0 = \oint_C f(z)dz - \sum_{i}^{n} \oint_{C_i} f(z)dz \tag{30}$$

This means that :

$$\oint_C f(z)dz = \sum_{i}^{n} \oint_{C_i} f(z)dz \tag{31}$$

Now performing Laurent series about each point z_i we have:

$$f(z) = \sum_{m=-\infty}^{\infty} a_m^{(i)} (z - z_i)^m \quad \text{about the point } z_i \tag{32}$$

and plugging it in the integrals:

$$\sum_{i=1}^{n} \oint_{C_i} \sum_{m=-\infty}^{\infty} a_m^{(i)} (z - z_i)^m dz \tag{33}$$

Now parametrizing $z - z_i = \epsilon_i e^{i\theta_i}$ and hence $dz = \epsilon_i i e^{i\theta_i} d\theta_i$:

$$\begin{aligned} & \sum_{i=1}^{n} \sum_{m=-\infty}^{\infty} a_m^{(i)} \int_0^{2\pi} \epsilon_i^m e^{im\theta_i} \epsilon_i e^{i\theta_i} i d\theta_i \\ & = \sum_{i=1}^{n} a_{-1}^{(i)} 2\pi i \end{aligned} \tag{34}$$

where the above integral is zero when $m \neq -1$ and the only contribution comes from the $m = -1$ term. This theorem is of extreme importance since one can do several integrals simply by using this formula.

Calculation of Residues

If $f(z)$ has a simple pole at $z = z_0$, then we have the Laurent series expansion about that point as:

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \ldots \tag{35}$$

Hence the coefficient a_{-1} can be computed as:

$$\begin{aligned} a_{-1} &= \lim_{z \to z_0} (z - z_0) f(z) \\ &= \lim_{z \to z_0} (z - z_0) [\frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \ldots] \\ &= \lim_{z \to z_0} [a_{-1} + a_0 (z - z_0) + a_1 (z - z_0)^2 + \ldots] \end{aligned} \tag{36}$$

All the terms apart from the first term go to zero. If $f(z)$ has a pole of order m at $z = z_0$, then we have the Laurent series :

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1 (z - z_0) + \ldots \tag{37}$$

Multiplying by $(z - z_0)^m$ we have :

$$f(z)(z - z_0)^m = a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + a_0 (z - z_0)^m + \ldots \tag{38}$$

Now we take the $(m-1)^{th}$ derivative :

$$\frac{d^{m-1}}{dz^{m-1}}[f(z)(z-z_0)^m] = 0 + \cdots + a_{-1}(m-1)! + a_0 m!(z-z_0) + \dots \tag{39}$$

Now if we take the $z \to z_0$ we obtain, since the remaining $z - z_0$ dependent terms go to zero :

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[f(z)(z-z_0)^m] = a_{-1}(m-1)! \tag{40}$$

From the above we can compute a_{-1}. Thus it is one's choice how to compute the residue, although performing the Laurent series is the most convenient way to do it.

Additional Theorems - Problems

17. JORDON' LEMMA : We have semi-circular contour C_R, of radius R, in the upper half complex plane ($z = x + iy$, $y > 0$), with center at the origin. Let $f(z)$ be a function that tends uniformly to zero with respect to $\theta = \arg z$ as $|z| \to \infty$ for $0 \leq \theta \leq \pi$. Then for $\alpha \in R_+$ we have :

$$\lim_{R \to \infty} \int_{C_R} e^{i\alpha z} f(z) dz = 0 \tag{41}$$

Now the fact that $f(z)$ tends to zero uniformly implies that, for any $0 \leq \theta \leq \pi$, we have on C_R :

$$|f(Re^{i\theta}| < \epsilon(R) \quad \text{such that} \quad \lim_{R \to \infty} \epsilon(R) \to 0 \tag{42}$$

Now let us look at the integral where $z = Re^{i\theta}$, since it is a semi-circle :

$$\begin{aligned} &\int_0^\pi iRe^{i\theta} d\theta f(Re^{i\theta}) e^{i\alpha(R\cos\theta + iR\sin\theta)} \\ &= \int_0^\pi iRe^{i\theta} d\theta f(Re^{i\theta}) e^{i\alpha R\cos\theta - \alpha R\sin\theta} \end{aligned} \tag{43}$$

The modulus of the above integral is obviously less than the integral of the modulus of the integrand (since then the integrand is always positive and hence there is no cancellation from elsewhere). This is obviously identical to the trinagle inequality if we interpret it as a Riemann Integral and write it as a sum. Thus :

$$\left| \int_0^\pi iRe^{i\theta} d\theta f(Re^{i\theta}) e^{i\alpha R\cos\theta - \alpha R\sin\theta} \right| \leq \int_0^\pi R d\theta |f(Re^{i\theta})| e^{-R\alpha \sin\theta} \tag{44}$$

Now using the limit on $|f(Re^{i\theta})|$

$$\int_0^\pi R d\theta |f(Re^{i\theta})| e^{-R\alpha\sin\theta} < \int_0^\pi R d\theta \epsilon(R) e^{-R\alpha\sin\theta} \tag{45}$$

Without going into any more rigorous analysis we see that taking the limit $R \to \infty$:

$$\lim_{R\to\infty} \int_0^\pi R d\theta \epsilon(R) e^{-R\alpha\sin\theta} \to 0 \tag{46}$$

This is because the exponential decays faster than R since in the upper half plane $\alpha \sin\theta > 0$, if $\epsilon(R)$ vanishes smoothly. We will use this lemma in several places to perform complex integrals.

18. Expand $\ln\left(\frac{1+z}{1-z}\right)$ in a Taylor series about $z = 0$ and check the convergence of the series by the ratio test :

$$\begin{aligned} &\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \quad \text{converges} \\ &\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1 \quad \text{diverges} \\ &\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1 \quad \text{inconclusive} \end{aligned} \tag{47}$$

19. The Legendre polynomials $P_n(t)$, $n = 0, 1, 2, 3, \ldots$ are given by the Rodrigues' formula :

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n \tag{48}$$

Then show that if C is a simple closed curve enclosing the point $z = t$ then :

$$P_n(t) = \frac{1}{2\pi i}\frac{1}{2^n}\oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} dz \tag{49}$$

Also prove that:

$$P_n(t) = \frac{1}{2\pi}\int_0^{2\pi} (t + \sqrt{t^2-1}\cos\theta)^n d\theta \tag{50}$$

20. Check the convergence of the series :

$$\frac{z}{3} + \frac{z(3-z)}{3^2} + \frac{z(3-z)^2}{3^3} + \frac{z(3-z)^3}{3^4} + \ldots \tag{51}$$

21. Perform the Taylor series of the following functions and check its domain of convergence:

$$\begin{aligned} &a)\frac{\sin z}{z^2+4} \qquad \text{about } z = 0 \\ &b) e^{-z^2}\sinh(z+2) \quad \text{about } z = 0 \\ &c) \sec \pi z \qquad \text{about } z = 1 \\ &d)\frac{e^z}{z(z-1)} \qquad \text{about } z = 4i \end{aligned} \tag{52}$$

22. Perform the Taylor series expansion of $\ln(3-iz)$ in powers of $z-2i$. Choose the "Principal branch" of the logarithm which means $\ln z = \ln r + i\theta + i2n\pi$ and n is taken to be zero. Cehck the convergence of the series.
23. Find the Laurent series of $f(z) = z/(z^2+1)$ for $|z-3| > 2$.
24. Find the Laurent series of $f(z) = 1/(z-2)^2$ for $|z| < 2$ and $|z| > 2$.
25. Expand each of the given functions about $z = 0$, mentioning the type of singularity:

$$a)\frac{(1-\cos z)}{z} \quad b)\frac{e^{z^2}}{z^3} \quad c)z^{-1}\cosh\frac{1}{z} \quad d)z^2 e^{-z^4} \quad e)z\sinh\sqrt{z} \tag{53}$$

26. Find the Laurent series of the following functions about $z = 0$. Also find all their singular points and classify the singularities:

$$a)\frac{1}{(2\sin z-1)^2} \quad b)\cos(z^2+z^{-2}) \tag{54}$$

27. Resum the following series in the region where it converges to a function which is well defined beyond the region of convergence:

$$a)\frac{1}{1+i}\sum_{n=0}^{\infty}\left(\frac{z+i}{1+i}\right)^n \quad b)\sum_{n=0}^{\infty}\frac{z^{n+1}}{3^n} \tag{55}$$

This is an example of "Analytic Continuation".
28. Prove Cauchy's inequality which states that if $f(z)$ is analytic inside and on a circle C of radius r and centre at $z = z_0$ then :

$$|f^{(n)}(z_0)| \leq \frac{M\,n!}{r^n} \quad n = 0, 1, 2, \ldots \tag{56}$$

where $|f(z)| < M$ on C.
28. Find the residues of $a)$ $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ and $b)$ $f(z) = e^z \mathrm{cosec}^2 z$ at all its poles in the finite complex plane.
29. Find the residues of $\frac{\cot z\,\mathrm{coth} z}{z^3}$ at $z = 0$
30. Evaluat the following integrasl using residue theorem:

$$\begin{aligned} &a)\int_0^\infty \frac{dx}{x^6+1} \\ &b)\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} \\ &c)\int_0^\infty \frac{x^{p-1}}{1+x}dx \quad 0 < p < 1 \\ &d)\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx \end{aligned} \tag{57}$$

31. Let us do an example and find the Fourier transform of $\frac{1}{x^2+a^2}$, where $a \in R$. This is defined as :

$$\int_{-\infty}^{\infty} \left(\frac{1}{x^2+a^2}\right) e^{-i2\pi x\zeta} dx \tag{58}$$

We do this in two parts : Firstly for $\zeta < 0$. For this we take the following contour Γ which is composed of two parts : C_1 which runs along the real axis from $-R$ to R and C_2 which is a semi circular anti-clockwise contour in the upper half plane, centered at origin and of radius R. In that case we have :

$$\oint_{\Gamma} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} = \oint_{\Gamma} \frac{dze^{-i2\pi\zeta z}}{(z+ia)(z-ia)} = \int_{C_1} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} + \int_{C_2} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} \tag{59}$$

Evaluating the residue, noting that there is only one pole in the upper half plane at $z = ia$, we have:

$$\begin{aligned}
&\oint_{\Gamma} \frac{dze^{-i2\pi\zeta z}}{(z+ia)(z-ia)} \\
&= 2\pi i \frac{e^{-i2\pi i\zeta(ia)}}{2ia} \\
&= \frac{\pi}{a} e^{2\pi\zeta a} \quad \text{Note that for } \zeta < 0, \text{ we have } \zeta = -|\zeta| \\
&= \frac{\pi}{a} e^{-2\pi|\zeta|a}
\end{aligned} \tag{60}$$

Next look at the contour C_2 :

$$\begin{aligned}
&\int_{C_2} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} = \int_0^{\pi} \frac{Rie^{i\theta} d\theta e^{-i2\pi\zeta(R\cos\theta + iR\sin\theta)}}{R^2e^{2i\theta}+a^2} \\
&= \int_0^{\pi} \frac{Rie^{i\theta} d\theta e^{i2\pi|\zeta|R\cos\theta - 2\pi|\zeta|R\sin\theta}}{R^2e^{2i\theta}+a^2}
\end{aligned} \tag{61}$$

The above goes to zero in the $R \to \infty$ limit by Jordon's lemma since $\sin\theta > 0$ in the upper half plane i.e. $0 \le \theta \le \pi$. Hence the remaining integral over C_1 becomes :

$$\begin{aligned}
&\int_{C_1} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx\, e^{-i2\pi\zeta x}}{x^2+a^2} \\
&= \int_{-\infty}^{\infty} \frac{dx\, e^{-i2\pi\zeta x}}{x^2+a^2}
\end{aligned} \tag{62}$$

which is the integral we needed. Now let us look at the other case : $\zeta > 0$. In this case the contour Γ is composed of two parts C_1 which is identical to previous case and C_3 which is a semi circular but clock-wise contour in the lower half plane, centered at origin and of radius R. Thus we have :

$$\oint_{\Gamma} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} = \oint_{\Gamma} \frac{dze^{-i2\pi\zeta z}}{(z+ia)(z-ia)} = \int_{C_1} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} + \int_{C_3} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} \tag{63}$$

Evaluating the residue and remembering that now the contour is clockwise, and there is only one pole inside the contour at $z = -ia$:

$$\begin{aligned}
&\oint_\Gamma \frac{dz e^{-i2\pi\zeta z}}{(z+ia)(z-ia)} \\
&= -2\pi i \frac{e^{-i2\pi i(-ia)}}{-2ia} \\
&= \frac{\pi}{a} e^{-2\pi|\zeta|a} \qquad \zeta > 0 \text{ hence } |\zeta| = \zeta
\end{aligned} \tag{64}$$

Next look at the contour C_3 :

$$\begin{aligned}
\int_{C_3} \frac{dz\, e^{-i2\pi\zeta z}}{z^2+a^2} &= \int_0^{-\pi} \frac{Rie^{i\theta} d\theta e^{-i2\pi\zeta(R\cos\theta + iR\sin\theta)}}{R^2 e^{2i\theta} + a^2} \\
= \int_0^{-\pi} \frac{Rie^{-i\theta} d\theta e^{-i2\pi|\zeta|R\cos\theta + 2\pi|\zeta|R\sin\theta}}{R^2 e^{2i\theta} + a^2}
\end{aligned} \tag{65}$$

But now this vanishes again in the $R \to \infty$ limit by Jordon's lemma since $\sin\theta < 0$ in the lower half plane i.e. $-\pi \le \theta \le 0$. Hence the remaining integral over C_1 again becomes:

$$\int_{-\infty}^{\infty} \frac{dx\, e^{-i2\pi\zeta x}}{x^2+a^2} \tag{66}$$

Thus we find the Fourier transform valid for all ζ:

$$\int_{-\infty}^{\infty} \frac{dx\, e^{-i2\pi\zeta x}}{x^2+a^2} = \frac{\pi}{a} e^{-2\pi|\zeta|a} \tag{67}$$

32 Show that, if $f(z)$ be analytic inside a closed contour C, except for finite number of poles inside C, then :

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P \tag{68}$$

Where N counts the number of zeroes along with multiplicity and P counts the number of poles with multiplicity.
33 Prove the following identity for $a > 0$:

$$\begin{aligned}
\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} y^s \frac{ds}{s} \qquad &= 0, \quad 0 < y < 1 \\
&= 1, \quad y > 1
\end{aligned} \tag{69}$$

34 If the Fourier transform is interpreted as a contour integral along the imaginary axis then under the change of variables : $t = \ln x$ and $ik = s$, we have:

$$\int_{-\infty}^{\infty} f(t) e^{-itk} dt \to \int_0^{\infty} \tilde{f}(x) x^{-s-1} dx, \quad \text{where} f(t) = f(\ln x) = \tilde{f}(x) \tag{70}$$

This is called the Melin transform. The inverse Fourier transform becomes:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikt}dk \to \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\phi(s)x^s ds, \quad a>0, \quad \text{where}\hat{f}(k)=\hat{f}(-is)=\phi(s) \tag{71}$$

Where a is the real part of s and is taken greater than zero. In most cases this is just ϵ distance away from the imaginary axis, but in some cases where $\phi(s)$ has singularities to the right of the imaginary axis the contour is taken such that the singularities are to the left of the contour i.e. $a > Re[s_0]$, such that $\lim_{s\to s_0}\phi(s)\to\infty$. Now use the above to find the counting function :

$$j_p(x)=\sum_{n=1}^{\infty}\delta(x-p^n) \tag{72}$$

for $p\in$ prime .

35. For a function $f(z)$, which decays as follows $|f(z)|\le\frac{M}{|z|^k}$, where $k>1$, for some given constant M, Prove that:

$$\sum_{n=-\infty}^{\infty}f(n) = -(\text{sum of residues of } (\pi\cot\pi z)f(z) \text{ at the poles of } f(z)) \tag{73}$$

To do this use a rectangular contour with vertices at $(N+\frac{1}{2})(1+i)$, $(N+\frac{1}{2})(-1+i)$, $(N+\frac{1}{2})(-1-i)$, $(N+\frac{1}{2})(1-i)$, evaluate the residues due to poles of $\cot\pi z$ and $f(z)$ and then use Darboux inquality to show that the contour integral vanishes in the $N\to\infty$ limit, hence obtaining a relation between the sum of the residues.

36. Hence prove:

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi}{a}\text{coth}\pi a, \quad a>0 \tag{74}$$

37. Prove the Mittag-Leffler expansion theorem which states that : given a function $f(z)$, having only simple poles at $a_1, a_2, \ldots$ in the the complex plane with residues $b_1, b_2, \ldots$ at these points respectively, we can write $f(z)$ as:

$$f(z)=f(0)+\sum_{n=1}^{\infty}b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right) \tag{75}$$

To prove this, a circle of radius R is taken, which does not pass through any of the poles, on which $|f(z)|<M$, and then limit $R\to\infty$ is taken. Hint : calculate the residues of $f(\zeta)/(\zeta-z)$ first.

38. Given an integral on the real axis with $z\in C$, and x is understood as a limit to the real axis:

$$H(z)=\int_a^b\frac{\rho(x)dx}{z-x} \tag{76}$$

Find the real and imaginay parts of the above function if z is taken to the limit of a point in the interval $a \le z \le b$. Note that there is a point of singularity in the integral.
39 Now given

$$H(h) = \ln\left[\frac{h + a - \sqrt{(h+a)^2 - bh}}{c}\right] \tag{77}$$

Find the real and imaginary parts of the above on the real axis, pointing out possible regions of singularity.
40 Given the Generating functions find the integral representations :

$$\begin{aligned} e^{\frac{x}{2}(t-\frac{1}{t})} &= \sum_{n=-\infty}^{\infty} J_n(x) t^n \\ e^{-t^2+2tx} &= \sum_{n=0}^{\infty} H_n(x) t^n \\ (1-2xt+t^2)^{-1/2} &= \sum_{n=0}^{\infty} P_n(x) t^n \end{aligned} \tag{78}$$

41Show that :

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \frac{2\pi}{(a^2-b^2)^{1/2}}, \quad |a| > |b| \tag{79}$$

42 Show that :

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\cos\theta t + t^2} = \frac{2\pi}{1-t^2}, \quad |t| < 1 \tag{80}$$

43 Show that :

$$\int_0^{\pi} \cos^{2n}\theta d\theta = \pi \frac{(2n)!}{2^{2n}(n!)^2} \tag{81}$$

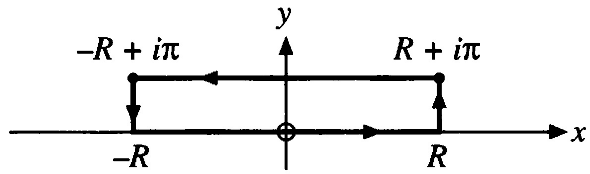
44 Evaluate :

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2} dx \tag{82}$$

To evaluate this integral perform the change of variables $x = e^z$:

$$\int_{-\infty}^{\infty} \frac{z^2}{1+e^{2z}} e^z dz \tag{83}$$

Now taking the following contour Γ :



Then we can evaluate the integral using the residue theorem. Looking at the function, we find it has simple poles at $2z = i(2n+1)\pi$ for $n = 0, \pm 1, \pm 2, \dots$, but only one of them lies inside this contour $z = i\frac{\pi}{2}$. Hence the residue is given by :

$$\lim_{z \to \frac{\pi}{2}} (z - i\frac{\pi}{2}) \frac{z^2}{1+e^{2z}} e^z = \frac{(i\frac{\pi}{2})^2 e^{i\frac{\pi}{2}}}{2e^{i\pi}} = i\frac{\pi^2}{8} \tag{84}$$

Now we decompose the contour Γ into parts :

$$\int_{-R}^{R} \frac{z^2}{1+e^{2z}} e^z dz + \int_{R}^{R+i\pi} \frac{z^2}{1+e^{2z}} e^z dz + \int_{R+i\pi}^{-R+i\pi} \frac{z^2}{1+e^{2z}} e^z dz + \int_{-R+i\pi}^{-R} \frac{z^2}{1+e^{2z}} e^z dz \tag{85}$$

The second and the last integral goes to zero in the $R \to \infty$ limit by Darboux inequality :

$$\left| \int_{R}^{R+i\pi} \frac{z^2}{1+e^{2z}} e^z dz \right| \le M\pi \tag{86}$$

and as

$$\lim_{R\to\infty} \frac{z^2}{1+e^{2z}} e^z \bigg|_{z=R+iy} = \lim_{R\to\infty} \frac{(R+iy)^2}{1+e^{2R+i2y}} e^{R+iy} \to 0 \quad \therefore M = 0 \tag{87}$$

Thus the integral vanishes. Similarly for the last term :

$$\lim_{R\to\infty} \frac{z^2}{1+e^{2z}} e^z \bigg|_{z=-R+iy} = \lim_{R\to\infty} \frac{(-R+iy)^2}{1+e^{-2R+i2y}} e^{-R+iy} \to 0 \tag{88}$$

Thus the first integral in the $R \to \infty$ becomes (replacing $z = x$):

$$\int_{-\infty}^{\infty} \frac{x^2}{1+e^{2x}} e^x dx \tag{89}$$

and the third term (replacing $z = x + i\pi$):

$$\begin{aligned} &\int_{\infty}^{-\infty} \frac{(x+i\pi)^2}{1+e^{2x+i2\pi}} e^{x+i\pi} dx \\ &= -\int_{\infty}^{-\infty} \frac{(x^2+2\pi i x-\pi^2)}{1+e^{2x}} e^x dx \\ &= \int_{-\infty}^{\infty} \frac{(x^2+2\pi i x-\pi^2)}{1+e^{2x}} e^x dx \\ &= \int_{-\infty}^{\infty} \frac{x^2}{1+e^{2x}} e^x dx + \int_{-\infty}^{\infty} \frac{2\pi i x}{e^{-x}+e^x} dx - \int_{-\infty}^{\infty} \frac{\pi^2}{1+e^{2x}} e^x dx \end{aligned} \tag{90}$$

The second integral is odd and hence is zero. The third integral can be evaluated as :

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\pi^2}{1+e^{2x}}e^x dx\\&=\int_0^{\infty}\frac{dt}{1+t^2}\\&=\tan^{-1}t\Big|_0^{\infty}=\frac{\pi}{2}\end{aligned} \tag{91}$$

Plugging this in we obtain in the complete integral over Γ :

$$\begin{aligned}&2\int_{-\infty}^{\infty}\frac{x^2}{1+e^{2x}}e^x dx-\frac{\pi^3}{2}=2\pi i(i\frac{\pi^3}{8})\\&\Rightarrow\int_{-\infty}^{\infty}\frac{x^2}{1+e^{2x}}e^x dx=\frac{\pi^3}{4}-\frac{\pi^3}{8}=\frac{\pi^3}{8}\end{aligned} \tag{92}$$

45 Let us evaluate the following integral:

$$\int_a^b\frac{\sqrt{(x-a)(b-x)}}{x(x^2+c^2)}dx \tag{93}$$

where $a,b,c\in\Re$. We start by looking at the following complex integral :

$$\oint_\Gamma\frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz \tag{94}$$

Where Γ is a contour around the Branch cut from a to b in the clock wise direction. The contour is decomposed into 4 parts :

$$\begin{aligned}&\int_{a+\epsilon}^{b-\epsilon}\frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz\Big|_{z=x+i\delta,\delta\to0_+}+\int_{\pi}^{-\pi}\frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz\Big|_{z=b+\epsilon e^{i\theta}}\\&+\int_{b-\epsilon}^{a+\epsilon}\frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz\Big|_{z=x-i\delta,\delta\to0_+}+\int_{2\pi}^{0}\frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz\Big|_{z=a+\epsilon e^{i\theta}}\end{aligned} \tag{95}$$

Where the first and third integrals are defined on a line parallel to the x axis but $\pm i\delta$ away from it to not lie on the branch cut. Let us look at the integrand in the first term :

$$\begin{aligned}&\lim_{\delta\to0_+}\frac{\sqrt{(x-a+i\delta)(x-b+i\delta)}}{(x+i\delta)((x+i\delta)^2+c^2)}\\&=\lim_{\delta\to0_+}\frac{\sqrt{|x-a+i\delta||x-b+i\delta|e^{i\tan^{-1}\frac{\delta}{x-a}+i\tan^{-1}\frac{\delta}{x-b}}}}{(x+i\delta)((x+i\delta)^2+c^2)}\\&=\lim_{\delta\to0_+}\frac{\sqrt{|x-a+i\delta||x-b+i\delta|}e^{\frac{i}{2}\tan^{-1}\frac{\delta}{x-a}+\frac{i}{2}\tan^{-1}\frac{\delta}{x-b}}}{(x+i\delta)((x+i\delta)^2+c^2)}\end{aligned} \tag{96}$$

The phase due to $z-a$ goes to zero while note that $x-b$ is negative since here $x<b$, hence :

$$\lim_{\delta\to 0_+}\tan^{-1}\frac{\delta}{x-b}=\lim_{\delta\to 0_+}\tan^{-1}\frac{-\delta}{b-x}=\lim_{\delta\to 0_+}\pi-\tan^{-1}\frac{\delta}{b-x}=\pi \tag{97}$$

which gives the contribution to the phase due to $z-b$ which gives for the first integrand :

$$\frac{\sqrt{|x-a||x-b|}e^{i\frac{\pi}{2}}}{x(x^2+c^2)}=\frac{\sqrt{(x-a)(b-x)}e^{i\frac{\pi}{2}}}{x(x^2+c^2)}\quad a<x<b \tag{98}$$

After this we go clockwise around point $z=b$, as is implied by the second integral. The third integrand then yields :

$$\begin{aligned}&\lim_{\delta\to 0_+}\frac{\sqrt{(x-a-i\delta)(x-b-i\delta)}}{(x-i\delta)((x-i\delta)^2+c^2)}\\&=\lim_{\delta\to 0_+}\frac{\sqrt{|x-a-i\delta||x-b-i\delta|}e^{\frac{i}{2}\tan^{-1}\frac{-\delta}{x-a}+\frac{i}{2}\tan^{-1}\frac{-\delta}{x-b}}}{(x-i\delta)((x-i\delta)^2+c^2)}\end{aligned} \tag{99}$$

The phase due to $z-a$ again is zero, since while traversing around the contour, we have not gone around a and the angle subtended by the point $z-a$ at a remains zero. While now for $z-b$ there is a change since while traversing the contour we have gone around b from π to $-\pi$ and hence the angle subtended by the point $z-b$ at b is :

$$\lim_{\delta\to 0_+}\tan^{-1}\frac{-\delta}{x-b}=\lim_{\delta\to 0_+}-\pi+\tan^{-1}+\frac{\delta}{b-x}=-\pi \tag{100}$$

Please understand the concept of taking the limit carefully, thus pointing out again :

$$\begin{aligned}&\tan^{-1}\frac{y}{x}\quad y>0,x<0,\ \text{lies in the second quadrant thus angle }\frac{\pi}{2}<\tan^{-1}\frac{y}{x}<\pi\\&\tan^{-1}\frac{y}{x}\quad y<0,x<0,\ \text{lies in the third quadrant thus angle }-\pi<\tan^{-1}\frac{y}{x}<-\frac{\pi}{2}\end{aligned} \tag{101}$$

In our specific case , for each of the integrands, while taking the limit , we approached the negative x axis from the second quadrant and third quadrant respectively, yielding π and $-\pi$. Coming back to the integrand for the third term we have: :

$$\frac{\sqrt{|x-a||x-b|}e^{-i\frac{\pi}{2}}}{x(x^2+c^2)}=\frac{\sqrt{(x-a)(b-x)}e^{-i\frac{\pi}{2}}}{x(x^2+c^2)}\quad a<x<b \tag{102}$$

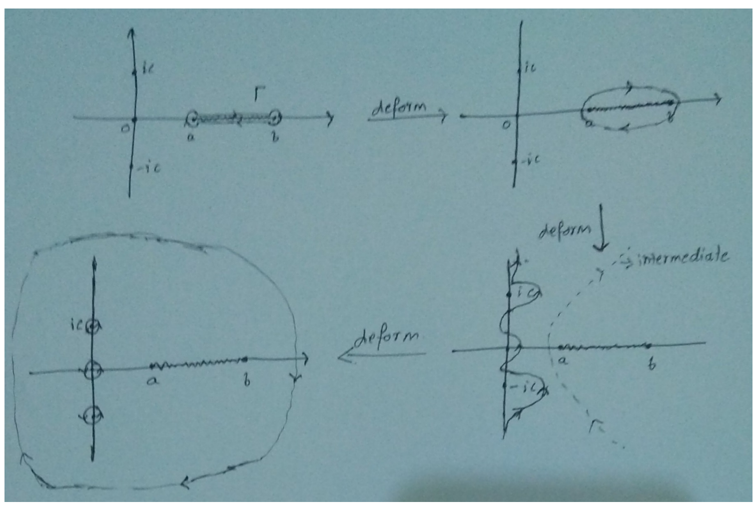
Let us now look at the second and fourth integrals. The second term is, replacing $z-b=\epsilon e^{i\theta}$

$$\int_{\pi}^{-\pi}\frac{\sqrt{(b-a+\epsilon e^{i\theta})\epsilon e^{i\theta}}}{(b+\epsilon e^{i\theta})((b+\epsilon e^{i\theta})^2+c^2)}i\epsilon e^{i\theta}d\theta \tag{103}$$

Taking the limit $\epsilon \to 0$ we see the integral vanishes. The fourth integral is also evaluated to zero analogously. Collecting all the terms after taking $\epsilon \to 0$ we have:

$$\begin{aligned}
&\oint_\Gamma \frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz = \int_a^b \frac{\sqrt{(x-a)(b-x)}e^{i\frac{\pi}{2}}}{x(x^2+c^2)}dx + \int_b^a \frac{\sqrt{(x-a)(b-x)}e^{-i\frac{\pi}{2}}}{x(x^2+c^2)}dx \\
\Rightarrow &\oint_\Gamma \frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz = 2i\int_a^b \frac{\sqrt{(x-a)(b-x)}}{x(x^2+c^2)}dx \\
\Rightarrow &\frac{1}{2i}\oint_\Gamma \frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz = \int_a^b \frac{\sqrt{(x-a)(b-x)}}{x(x^2+c^2)}dx
\end{aligned} \tag{104}$$

Thus we have converted the integral we wanted into a complex contour integration. This can be evaluated using Residue theorem. Now deform Γ by making it larger and larger. It would start looking like a bubble. While increasing it we will encounter points of singularity at $z = 0, \pm ic$. The contour will then go around thee points , slowly engulfing them. Then we can expand the contour out to infinity. Thus we are left with a clockwise integral at infinity and counter-clockwise integral around the points of singularity.



$$\begin{aligned}
\oint_\Gamma \frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz = &\lim_{R\to\infty}\int_{2\pi}^{0} \frac{\sqrt{(Re^{i\theta}-a)(Re^{i\theta}-b)}}{Re^{i\theta}(R^2e^{i2\theta}+c^2)}Re^{i\theta}id\theta \\
&+ \sum_{z_i=0,\pm ic}\oint_{c_i} \frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz\Big|_{z=z_i+\epsilon e^{i\theta}}
\end{aligned} \tag{105}$$

Where c_i are anti-clockwise contours around the points of singularity. The first integral is zero as the integrand goes to zero like $1/R^2$ in the $R \to \infty$ limit (rigorously seen by Darboux inquality/Jordon's Lemma). The remaining can be evaluated by residue theorem. Residue at $z = 0$:

$$\lim_{z\to 0} z\frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)} = \frac{\sqrt{ab\, e^{i2\pi}}}{c^2} = -\frac{\sqrt{ab}}{c^2} \tag{106}$$

Residue due to $z = ic$:

$$\lim_{z\to ic}(z-ic)\frac{\sqrt{(z-a)(z-b)}}{z(z-ic)(z+ic)} = \frac{\sqrt{(-a+ic)(-b+ic)}}{ic(i2c)}$$
$$= \frac{(a^2+c^2)^{\frac{1}{4}}(b^2+c^2)^{\frac{1}{4}}e^{\frac{i}{2}\tan^{-1}\frac{c}{-a}+\frac{i}{2}\tan^{-1}\frac{c}{-b}}}{-2c^2}$$
$$= -\frac{(a^2+c^2)^{\frac{1}{4}}(b^2+c^2)^{\frac{1}{4}}}{2c^2}(-\cos(\frac{1}{2}\tan^{-1}\frac{c}{a}+\frac{1}{2}\tan^{-1}\frac{c}{b})+i\sin(\frac{1}{2}\tan^{-1}\frac{c}{a}+\frac{1}{2}\tan^{-1}\frac{c}{b})) \tag{107}$$

Similarly evaluating the residue at $z = -ic$ carefully yileds :

$$= -\frac{(a^2+c^2)^{\frac{1}{4}}(b^2+c^2)^{\frac{1}{4}}}{2c^2}(-\cos(\frac{1}{2}\tan^{-1}\frac{c}{a}+\frac{1}{2}\tan^{-1}\frac{c}{b})-i\sin(\frac{1}{2}\tan^{-1}\frac{c}{a}+\frac{1}{2}\tan^{-1}\frac{c}{b})) \tag{108}$$

Thus adding all three, the imaginary parts cancel yielding :

$$\oint_\Gamma \frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz = 2\pi i\left[\frac{(a^2+c^2)^{\frac{1}{4}}(b^2+c^2)^{\frac{1}{4}}}{c^2}\cos(\frac{1}{2}\tan^{-1}\frac{c}{a}+\frac{1}{2}\tan^{-1}\frac{c}{b})-\frac{\sqrt{ab}}{c^2}\right] \tag{109}$$

Thus finally we obtain :

$$\int_a^b \frac{\sqrt{(x-a)(b-x)}}{x(x^2+c^2)}dx = \frac{1}{2i}\oint_\Gamma \frac{\sqrt{(z-a)(z-b)}}{z(z^2+c^2)}dz$$
$$= \pi\left[\frac{(a^2+c^2)^{\frac{1}{4}}(b^2+c^2)^{\frac{1}{4}}}{c^2}\cos(\frac{1}{2}\tan^{-1}\frac{c}{a}+\frac{1}{2}\tan^{-1}\frac{c}{b})-\frac{\sqrt{ab}}{c^2}\right] \tag{110}$$

Such a complicated integral can be evaluated using the method of complex analysis. As a quick check we see that the integral must be zero in $a \to b$ limit :

$$\pi\left[\frac{(a^2+c^2)^{\frac{1}{4}}(a^2+c^2)^{\frac{1}{4}}}{c^2}\cos(\frac{1}{2}\tan^{-1}\frac{c}{a}+\frac{1}{2}\tan^{-1}\frac{c}{a})-\frac{a}{c^2}\right]$$
$$\Rightarrow \pi\left[\frac{\sqrt{a^2+c^2}}{c^2}\frac{a}{\sqrt{a^2+c^2}}-\frac{a}{c^2}\right] = 0 \tag{111}$$

Course Material on Fourier Transforms

Definition

A periodic function $f(x)$ in the interval $[-L, L]$ can be expanded in a fourier series as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} \tag{112}$$

This means the fourier coeffiecients can be given as :

$$\int_{-L}^{L} f(x) e^{-i\frac{m\pi x}{L}} dx = \int_{-L}^{L} \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} e^{-i\frac{m\pi x}{L}} = c_m 2L \tag{113}$$

Let us rename the fourier coefficient c_n as $\frac{1}{2L}\hat{f}(n/2L)$ and call $n/2L = \zeta_n$. Hence:

$$\hat{f}(n/2L) = \int_{-L}^{L} f(x) e^{-i\frac{n2\pi x}{2L}} dx, \quad f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L}\hat{f}(n/2L) e^{i\frac{2n\pi x}{2L}} \tag{114}$$

Note that $\triangle\zeta = \zeta_{n+1} - \zeta_n = \frac{n+1}{2L} - \frac{n}{2L} = \frac{1}{2L}$. Therefore:

$$\hat{f}(\zeta_n) = \int_{-L}^{L} f(x) e^{-i2\pi x\zeta_n} dx, \quad f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(\zeta_n) e^{i2\pi\zeta_n x} \triangle\zeta \tag{115}$$

In the $L \to \infty$ limit the sum on the R.H.S. can be interpreted as a Riemann sum and hence the above becomes ($\zeta_n \to \zeta$ a continuous parameter):

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x\zeta} dx \Rightarrow \text{Fourier Transform} \tag{116}$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{i2\pi\zeta x} d\zeta \Rightarrow \text{Inverse Fourier Transform} \tag{117}$$

Please note the extra factor of 2π in the exponential which is a convention here and an *additional - sign which is different from standard notation.*. More over just like fourier series we have the condition that, the integral $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, i.e. must converge.

Examples-Theorems

1. From the periodic delta function $\delta(x-a) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} e^{i\frac{n\pi}{L}(x-a)}$, derive the representation of the Dirac delta function in the limit $2L \to \infty$, in the same way as above by replacing $\zeta_n = \frac{n}{2L}$ and $\triangle\zeta = 1/2L$

$$\delta(x-a) = \int_{-\infty}^{\infty} e^{i2\pi\zeta(x-a)} \tag{118}$$

2. Show that the fourier invers of the delta function $\delta(x-a)$ is $e^{-i2\pi\zeta a}$.
3. Find fourier transform of the following functions:

$$\begin{aligned} &a)\ \frac{\delta(x-a)+\delta(x+a)}{2} \\ &b)\ \frac{\delta(x-a)-\delta(x+a)}{2} \end{aligned} \tag{119}$$

4. Show that the Fourier transform of real even function is real and Fourier transform of real odd function is purely imaginary.

5. CONVOLUTION THEOREM : Inverse Fourier transform of product of two functions is :

$$\begin{aligned} &\int_{-\infty}^{\infty} \hat{f}(\zeta)\hat{g}(\zeta)e^{i2\pi\zeta x}d\zeta \\ &= \int_{-\infty}^{\infty} \hat{f}(\zeta) \int_{-\infty}^{\infty} \delta(\zeta'-\zeta)\hat{g}(\zeta')e^{i2\pi\zeta x}d\zeta d\zeta' \\ &= \int_{-\infty}^{\infty} \hat{f}(\zeta) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i2\pi(\zeta'-\zeta)t}dt\, \hat{g}(\zeta')e^{i2\pi\zeta x}d\zeta d\zeta' \end{aligned} \tag{120}$$

where we have introduced the integral representation of the delta function above. Hence:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\zeta)e^{i2\pi\zeta(x-t)}d\zeta \int_{-\infty}^{\infty} e^{i2\pi\zeta' t}\hat{g}(\zeta')d\zeta'\, dt \\ &= \int_{-\infty}^{\infty} f(x-t)g(t)dt \end{aligned} \tag{121}$$

We have just used the definition of the Inverse Fourier Transform. Hence Fourier transform of product of two functions is not the product of the Fourier transforms of the individual ones. Generalize this for product of any number of functions.

6. PERSEVAL'S THEOREM : This has application in Quantum mechanics, interpreted as total position space probability = total momentum space probability, as the wave function can be written in any one of the "dual-basis".

$$\int_{-\infty}^{\infty} f(x)f^*(x)dx = \int_{-\infty}^{\infty} \hat{f}(\zeta)\hat{f}^*(\zeta)d\zeta \tag{122}$$

To prove this we observe that :

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{i2\pi\zeta x}d\zeta \tag{123}$$

and its complex conjugation is given by (please note that ζ and ζ' are dummy integration variables):

$$f^*(x) = \int_{-\infty}^{\infty} \hat{f}^*(\zeta)e^{-i2\pi\zeta' x}d\zeta' \tag{124}$$

Plugging both of them in we have :

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)f^*(x)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\zeta)e^{i2\pi\zeta x}d\zeta \int_{-\infty}^{\infty} \hat{f}^*(\zeta)e^{-i2\pi\zeta' x}d\zeta' dx \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta)d\zeta \int_{-\infty}^{\infty} e^{i2\pi(\zeta-\zeta')x}dx \int_{-\infty}^{\infty} \hat{f}^*(\zeta)d\zeta' \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta)d\zeta\, \delta(\zeta-\zeta') \int_{-\infty}^{\infty} \hat{f}^*(\zeta)d\zeta' \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta)\hat{f}^*(\zeta)d\zeta \qquad (125)
\end{aligned}$$

In Quantum mechanics $f(x)$ plays the role of the particle wave function. More over this proves that the Fourier transform is a Unitary transformation over the vector space of well behaved functions, since the "inner-product" which is the integral here, is left invariant. It is straight forward to generalize to the case of two different functions :

$$\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty} \hat{f}(\zeta)\hat{g}^*(\zeta)d\zeta \qquad (126)$$

7. DERIVATIVE : From the Inverse Fourier transform, we have :

$$\begin{aligned}
& f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{i2\pi\zeta x}d\zeta \\
& \Rightarrow f'(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)(i2\pi\zeta)e^{i2\pi\zeta x}d\zeta \\
& \Rightarrow f^{(n)}(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)(i2\pi\zeta)^n e^{i2\pi\zeta x}d\zeta
\end{aligned} \qquad (127)$$

Applying the Fourier transform on both side :

$$\int_{-\infty}^{\infty} f^{(n)}(x)e^{-i2\pi\zeta' x}dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\zeta)(i2\pi\zeta)^n e^{i2\pi(\zeta-\zeta')x}dx\, d\zeta \qquad (128)$$

Then using the representation of the delta function :

$$\int_{-\infty}^{\infty} f^{(n)}(x)e^{-i2\pi\zeta' x}dx = \hat{f}(\zeta')(i2\pi\zeta')^n \qquad (129)$$

8. POISSON SUMMATION FORMULA (Very Important in Number theory and resummation). For a regular function $f(x)$ we have:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \qquad (130)$$

The L.H.S. can be written as :

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) \sum_{n=-\infty}^{\infty} \delta(x-n)dx \tag{131}$$

We observe that the periodic delta function, with period 1 and located at integers, can be written as:

$$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{m=-\infty}^{\infty} e^{i2\pi xm} \tag{132}$$

as can be seen, $2L = 1$ here and $a = 0$ from the previous example for the delta function. Plugging this in we have :

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) \sum_{m=-\infty}^{\infty} e^{i2\pi xm}dx \tag{133}$$

Interchanging the sum and the integral, since the function is regular:

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)e^{i2\pi xm}dx \qquad \left(\text{by definition} \hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\zeta x}dx\right)$$

$$= \sum_{m-\infty}^{\infty} \hat{f}(-m)$$

now inverting the limit of the sum since the order does not matter if sum is absolutely convergent

$$= \sum_{m-\infty}^{\infty} \hat{f}(m) \tag{134}$$

This has several applications in summation and in number theory.

Preliminary Problems

9. Fourier transform of the Gaussian :

$$f(x) = e^{-\alpha x^2} \tag{135}$$

Hence its Fourier transform can be computed as :

$$\begin{aligned}
\hat{f}(\zeta) \quad &= \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-i2\pi\zeta x}dx \\
&= \int_{-\infty}^{\infty} e^{-\alpha x^2 - i\frac{2\pi\zeta x}{\sqrt{\alpha}}\sqrt{\alpha} + (i\frac{\pi\zeta}{\sqrt{\alpha}})^2 - (i\frac{\pi\zeta}{\sqrt{\alpha}})^2} dx \\
&= \int_{-\infty}^{\infty} e^{-(x\sqrt{\alpha} + i\frac{\pi\zeta}{\sqrt{\alpha}})^2 - (\frac{\pi\zeta}{\sqrt{\alpha}})^2} dx
\end{aligned} \tag{136}$$

Now we can redefine $x\sqrt{\alpha}+i\frac{\pi\zeta}{\sqrt{\alpha}}=t$. Then $\sqrt{\alpha}dx=dt$. Hence we have:

$$\frac{1}{\sqrt{\alpha}}\int_{-\infty+i\frac{\pi\zeta}{\sqrt{\alpha}}}^{\infty+i\frac{\pi\zeta}{\sqrt{\alpha}}} e^{-t^2}e^{-(\frac{\pi\zeta}{\sqrt{\alpha}})^2}dt \tag{137}$$

Now note that the t integral goes along a line which is parallel to the real axis in the complex plane with a imaginary shift by $+i\frac{\pi\zeta}{\sqrt{\alpha}}$, and hence it is not strictly the real axis (let us call this contour Γ). But now observe that by Cauchy's theorem, the function e^{-t^2} is analytic every where in the complex t plane. Hence one can shift the contour Γ to make it along x axis to another contour Γ' which goes from $-\infty$ to ∞. Thus we have:

$$e^{-(\frac{\pi\zeta}{\sqrt{\alpha}})^2}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-t^2}dt \tag{138}$$

Now the integral over t can be evaluated using the Gamma function and yields $\sqrt{\pi}$:

$$\sqrt{\frac{\pi}{\alpha}}e^{-(\frac{\pi\zeta}{\sqrt{\alpha}})^2} \tag{139}$$

This show Fourier transform of the Gaussian is another Gaussian function.
10. Perform the fourier trasform of the function $f(x)=e^{-a|x|}$.
11. The n^{th} energy eigen function of a quantum Harmonic oscillator satisfies the following time independent Schrodinger equation (we use $\hbar, m, k=1$):

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2\right]\phi_n(x)=(n+\frac{1}{2})\phi_n(x) \tag{140}$$

Where $\phi_n(x)=e^{-\frac{x^2}{2}}H_n(x)$, written in terms of the Hermite polynomials. If the momentum space wave function is defined by the fourier transform :

$$\hat{\phi}_n(\zeta)=\int_{-\infty}^{\infty}\phi_n(x)e^{-i2\pi\zeta x}dx \tag{141}$$

Show that this satisfies the same differential equation with x replaced by $2\pi\zeta$ and hence $\hat{\phi}_n=c_ne^{-\frac{(2\pi\zeta)^2}{2}}H_n(2\pi\zeta)$, where c_n are just some constants.
12. Perform the Fourier transform of $f(x)=\text{sech}(x)=\frac{2}{e^{ax}+e^{-ax}}$ first directly, where you can resum the series after the integral by the expansion:

$$\text{sech}z=\pi\left(\frac{1}{(\frac{\pi}{2})^2+z^2}-\frac{3}{(\frac{3\pi}{2})^2+z^2}+\frac{5}{(\frac{5\pi}{2})^2+z^2}-\dots\right) \tag{142}$$

and again by using *Residue theorem.*
13. Perform the Fourier transform of the square pulse $f(x)=1, \quad -a\leq x\leq a$. Hence perform its inverse to get the pulse back.

Interpretation and discussion

The Fourier transform just like the fourier series can be interpreted as superposition of waves of different frequencies. For example we see for the square pulse:

$$f(x) \sim \sum_{\zeta} \frac{\sin 2\pi\zeta a}{\pi\zeta} e^{-i2\pi\zeta x} \rightarrow \int_{-\infty}^{\infty} \frac{\sin 2\pi\zeta a}{\pi\zeta} e^{-i2\pi\zeta x} \tag{143}$$

The term $\frac{\sin 2\pi\zeta a}{\pi\zeta}$ can be interpreted as the amplitude of the wave $e^{-i2\pi\zeta x}$. These waves form an orthogonal basis in an infinite dimensional vector space. Let us remind ourselves the case for the fourie series :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{2L}x} \tag{144}$$

Here we have the orthogonal "vectors" in a countable discrete basis as:

$$\int_{-L}^{L} e^{i\frac{2\pi}{2L}(n-m)x} dx = 2L\delta_{mn} \tag{145}$$

Similarly we have :

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{i2\pi\zeta x} d\zeta \tag{146}$$

multiplying both side with $e^{-i2\pi\zeta' x}$ and integrating with respect to x

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) e^{-i2\pi\zeta' x} dx &= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{i2\pi\zeta x} e^{-i2\pi\zeta' x} d\zeta\, dx \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{i2\pi(\zeta-\zeta')x} dx\, d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) \delta(\zeta - \zeta') d\zeta \\
&= \hat{f}(\zeta') \quad \text{(coefficient corresponding to wave } e^{i2\pi\zeta x}\text{)}
\end{aligned} \tag{147}$$

Hence we have the orthogonal "vectors" in a continuum basis :

$$\int_{-\infty}^{\infty} e^{i2\pi\zeta x} e^{-i2\pi\zeta' x} dx = \delta(\zeta - \zeta') \rightarrow \text{limit of } 2L\delta_{nm} \tag{148}$$

Hence $e^{i2\pi\zeta x}$ forms a complete orthonormal basis over space of integrable functions ($\int_{-\infty}^{\infty} |f(x)| dx < \infty$).

Additional Problems

14. Fourier transform of x^n . Please not that this is a function which does not satisfy the integrability condition. We see that:

$$\begin{aligned}
\hat{f}(\zeta) &= \int_{-\infty}^{\infty} x^n e^{-i2\pi\zeta x} dx \\
&= \int_{-\infty}^{\infty} \frac{d^n}{d(-i2\pi\zeta)^n} e^{-i2\pi\zeta x} dx \\
&= \left(\frac{i}{2\pi}\right)^n \frac{d^n}{d\zeta^n} \int_{-\infty}^{\infty} e^{-i2\pi\zeta x} \\
&= \left(\frac{i}{2\pi}\right)^n \frac{d^n}{d\zeta^n} \delta(\zeta)
\end{aligned} \tag{149}$$

where we have used the representation of the delta function. Obviously the above makes sense only as a distribution (generalized function) and hence under integration with a smooth well behaved function.

15. Fourier transform of $f(x) = \text{sign}(x)$

$$\begin{aligned}
\hat{f}(\zeta) &= \int_{-\infty}^{\infty} \text{sign}(x) e^{-i2\pi\zeta x} dx \\
&= -\int_{-\infty}^{0} e^{-i2\pi\zeta x} dx + \int_{0}^{\infty} e^{-i2\pi\zeta x} dx \\
&= -\frac{e^{-i2\pi x\zeta}}{-i2\pi\zeta}\bigg|_{-\infty}^{0} + \frac{e^{-i2\pi x\zeta}}{-i2\pi\zeta}\bigg|_{0}^{\infty} \\
&= \frac{1}{i2\pi\zeta} - \lim_{\Lambda\to\infty} \frac{e^{i2\pi\Lambda\zeta}}{i2\pi\zeta} - \lim_{\Lambda\to\infty} \frac{e^{-i2\pi\Lambda\zeta}}{i2\pi\zeta} + \frac{1}{i2\pi\zeta} \\
&= \frac{1}{i\pi\zeta} - \lim_{\Lambda\to\infty} \frac{1}{i2\pi\zeta}(e^{i2\pi\Lambda\zeta} + e^{-i2\pi\Lambda\zeta})
\end{aligned} \tag{150}$$

The last term fluctuates wildly as $\Lambda \to \infty$. Hence integral of any smooth function $g(\zeta)$ with this will give zero as integral will have infinitely many full periods where $g(\zeta)$ is sufficiently constant. *Perform the problem again by doing integration by-parts.*

16. Perform the Fourier Transform of the Dirac Comb : $f(x) = \sum_{-\infty}^{\infty} \delta(x - nT)$

17. It is easy to generalize to higher dimensions, for example in two dimensions we have:

$$\hat{f}(\zeta_x, \zeta_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(\zeta_x x + \zeta_y y)} dx\, dy \tag{151}$$

18 Perform the Fourier transform of the Columb potential $1/|\vec{r}|$ in 3 dimensions.

19 Perform Fourier transform of the following function :

$$\begin{aligned}
f(x) &= \sin\omega x \quad |x| < \frac{N\pi}{\omega_0} \\
&= 0 \qquad |x| > \frac{N\pi}{\omega_0}
\end{aligned} \tag{152}$$

20 Show that :

$$\int_0^\infty \frac{\omega \sin \omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2} e^{-ax} \quad x > 0 \tag{153}$$

21 Perform the Fourier transform of :

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x^2 - a^2}}, \quad & |x| < a \\ &= 0 & |x| > a \end{aligned} \tag{154}$$

22 Find Fourier transform of :

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \cos(nx)(1 - x^2)^{-1/2}, \quad & |x| < 1 \\ &= 0 & |x| > 1 \end{aligned} \tag{155}$$

23 Use Perseval identity to evaluate :

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt \tag{156}$$

24 Given :

$$\begin{aligned} f(x) &= 1 - |x/2|, \quad & -2 \leq x \leq 2 \\ &= 0 & \text{elsewhere} \end{aligned} \tag{157}$$

find the Fourier transform of the above. Hence find :

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^4 dt \tag{158}$$

25 Find the value of the following integrals :

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2 + a^2)^2} \tag{159}$$

Application to solution of differential equations

The method of Fourier transform is also used to solve Partial Differential Equations.

- As an example we can look at the wave equation :

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2} \quad \text{initial conditions : } y(x,0) = f(x) \tag{160}$$

where x is the position coordinate and t is time, while v is the velocity of the wave. Applying Fourier transform on both sides :

$$\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial x^2} e^{-i2\pi x\zeta} dx = \int_{-\infty}^{\infty} \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2} e^{-i2\pi x\zeta} dx \tag{161}$$

using the fact that :

$$\int_{-\infty}^{\infty} f^{(n)}(x) e^{-i2\pi x\zeta} dx = \hat{f}(\zeta)(i2\pi\zeta)^2 \tag{162}$$

We have:

$$(i2\pi\zeta)^2 \int_{-\infty}^{\infty} y(x,t) e^{-i2\pi x\zeta} dx = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} y(x,t) e^{-i2\pi x\zeta} dx \tag{163}$$

where we have taken the derivative with respect to t outside the integral, since it is with respect to x. Denoting the Fourier transform of $y(x,t)$ as $\tilde{y}(\zeta,t)$, we obtain from above equation :

$$-(2\pi\zeta)^2 \tilde{y}(\zeta,t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \tilde{y}(\zeta,t) \tag{164}$$

This is now an ordinary differential equation and can be solved more easily. The solution of this is :

$$\tilde{y}(\zeta,t) = F(\zeta) e^{\pm iv2\pi\zeta t} \tag{165}$$

Where $F(\zeta)$ is fixed from the boundary condition as we will shortly see. Pluggin this into the inverse Fourier transform :

$$\begin{aligned} y(x,t) &= \int_{-\infty}^{\infty} \tilde{y}(\zeta,t) e^{i2\pi x\zeta} d\zeta \\ &= \int_{-\infty}^{\infty} F(\zeta) e^{\pm iv2\pi\zeta t} e^{i2\pi x\zeta} d\zeta \\ &= \int_{-\infty}^{\infty} F(\zeta) e^{i2\pi\zeta(x \pm vt)} d\zeta \end{aligned} \tag{166}$$

evidently for $t = 0$, we have the boundary condition :

$$y(x,0) = f(x) = \int_{-\infty}^{\infty} F(\zeta) e^{i2\pi\zeta x} d\zeta \tag{167}$$

Thus $F(\zeta)$ is the Fourier transform of $f(x)$.

- We look at the equation for flow of heat through a one dimensional medium :

$$\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2} \tag{168}$$

Where $\psi(x,t)$ gives the temperature profile in space (x) and time (t). We employ the same technique and perform a Fourier transform in space :

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\partial\psi(x,t)}{\partial t}e^{-i2\pi x\zeta}dx &= \int_{-\infty}^{\infty}a^2\frac{\partial^2\psi(x,t)}{\partial x^2}e^{-i2\pi x\zeta}dx\\ \Rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\psi(x,t)e^{-i2\pi x\zeta}dx &= a^2(i2\pi\zeta)^2\int_{-\infty}^{\infty}\psi(x,t)e^{-i2\pi x\zeta}dx\end{aligned} \tag{169}$$

Denoting the Fourier transform of $\psi(x,t)$ as $\tilde{\psi}(\zeta,t)$ we have :

$$\frac{\partial}{\partial t}\tilde{\psi}(\zeta,t) = -(a2\pi\zeta)^2\tilde{\psi}(\zeta,t) \tag{170}$$

This can be easily be solved with the solution given by :

$$\tilde{\psi}(\zeta,t) = \tilde{\psi}(\zeta,0)e^{-(2\pi a\zeta)^2 t} \tag{171}$$

Now performing the inverse Fourier transform on $\tilde{\psi}(\zeta,t)$:

$$\psi(x,t) = \int_{-\infty}^{\infty}\tilde{\psi}(\zeta,0)e^{-(2\pi a\zeta)^2 t}e^{i2\pi x\zeta}d\zeta \tag{172}$$

for a given boundary condition at initial temperature $\psi(x,0)$ we can solve the above exactly. As a special case if $\psi(x,0)=\delta(x)$, then :

$$\delta(x) = \int_{-\infty}^{\infty}\tilde{\psi}(\zeta,0)e^{i2\pi x\zeta}d\zeta \tag{173}$$

Which means $\tilde{\psi}(\zeta,0)=1$. Plugging this in :

$$\psi(x,t) = \int_{-\infty}^{\infty}e^{-4\pi^2a^2\zeta^2 t}e^{i2\pi x\zeta}d\zeta \tag{174}$$

Which is a Gaussian integral. **Perform the integral and find the final form.**

- Let us try to solve an Ordinary differential equation using this technique. Given the following :

$$-D\frac{d^2\phi(x)}{dx^2} + K^2D\phi(x) = Q\delta(x) \tag{175}$$

Performing fourier transform on both sides :

$$\begin{aligned}\int_{-\infty}^{\infty}(-D\frac{d^2\phi(x)}{dx^2} + K^2D\phi(x))e^{-i2\pi x\zeta}dx &= \int_{-\infty}^{\infty}Q\delta(x)e^{-i2\pi x\zeta}dx\\ \int_{-\infty}^{\infty}(-D(i2\pi\zeta)^2 + DK^2)\phi(x)e^{-i2\pi x\zeta}dx &= \int_{-\infty}^{\infty}Q\delta(x)e^{-i2\pi x\zeta}dx\end{aligned} \tag{176}$$

denoting $\tilde{\phi}(\zeta)$ as the Fourier transform of $\phi(x)$ we have :

$$\begin{aligned} D(4\pi^2\zeta^2 + K^2)\tilde{\phi}(\zeta) &= Q \\ \Rightarrow \tilde{\phi}(\zeta) &= \frac{Q}{D(4\pi^2\zeta^2 + K^2)} \end{aligned} \tag{177}$$

This is nothing but the Lorentzian whose Fourier we already computed. Thus performing the inverse:

$$\phi(x) = \int_{-\infty}^{\infty} \frac{Q}{D(4\pi^2\zeta^2 + K^2)} e^{i2\pi x\zeta} d\zeta = \frac{Q}{2KD} e^{-|Kx|} \tag{178}$$

using Residue theorem.

- An important part of this is the generalization of the above in higher dimensions. Thus we look at the following differential equation :

$$\nabla^2 G(\vec{r}, \vec{r}') = -\delta^3(\vec{r} - \vec{r}') \tag{179}$$

This plays role in solving Poisson's equation, as for example in electro-statics we have :

$$\Phi(\vec{r}) = \int G(\vec{r}, \vec{r}') 4\pi\rho(\vec{r}') d\, v' \tag{180}$$

applying ∇^2 , with respect to $\vec{r}$, on the above equation we obtain :

$$\begin{aligned} \nabla^2\Phi(\vec{r}) &= \int \nabla^2 G(\vec{r}, \vec{r}') 4\pi\rho(\vec{r}') d\, v' \\ &= -\int \delta^3(\vec{r} - \vec{r}') 4\pi\rho(\vec{r}') d\, v' \\ &= -4\pi\rho(\vec{r}) \end{aligned} \tag{181}$$

Now to find the greens function we write both sides of the equation in Fourier space :

$$\begin{aligned} G(\vec{r}, \vec{r}') &= \int e^{i2\pi(\vec{r}-\vec{r}').\vec{\zeta}} \tilde{G}(\vec{\zeta}) d^3\zeta \\ \delta^3(\vec{r} - \vec{r}') &= \int e^{i2\pi(\vec{r}-\vec{r}').\vec{\zeta}} d^3\zeta \end{aligned} \tag{182}$$

Plugging this in to the differential equation for the Green's function :

$$\begin{aligned} \nabla^2 \int e^{i2\pi(\vec{r}-\vec{r}').\vec{\zeta}} \tilde{G}(\vec{\zeta}) d^3\zeta &= -\int e^{i2\pi(\vec{r}-\vec{r}').\vec{\zeta}} d^3\zeta \\ \int (i2\pi\vec{\zeta}).(i2\pi\vec{\zeta}) e^{i2\pi(\vec{r}-\vec{r}').\vec{\zeta}} \tilde{G}(\vec{\zeta}) d^3\zeta &= -\int e^{i2\pi(\vec{r}-\vec{r}').\vec{\zeta}} d^3\zeta \end{aligned} \tag{183}$$

Since the integrals are identical for all values of $\vec{r}-\vec{r}'$, we must have the integrands to be same, hence :

$$\begin{aligned} -4\pi^2|\vec{\zeta}|^2\tilde{G}(\vec{\zeta}) &= -1 \\ \Rightarrow \tilde{G}(\vec{\zeta}) &= \frac{1}{4\pi^2|\vec{\zeta}|^2} \end{aligned} \tag{184}$$

Plugging into the Fourier representation for the Green's function :

$$G(\vec{r},\vec{r}') = \int e^{i2\pi(\vec{r}-\vec{r}').\vec{\zeta}} \frac{1}{4\pi^2|\vec{\zeta}|^2} d^3\zeta \tag{185}$$

This integral can be performed in spherical polar coordinates $d^3\zeta = |\vec{\zeta}|^2 d|\vec{\zeta}| \sin\theta \, d\theta d\phi$, we have :

$$\begin{aligned} G(\vec{r},\vec{r}') &= \int_0^\infty \int_{-\pi}^{\pi} \int_0^{2\pi} e^{i2\pi|\vec{r}-\vec{r}'||\vec{\zeta}|\cos\theta} \frac{1}{4\pi^2|\vec{\zeta}|^2} |\vec{\zeta}|^2 d|\vec{\zeta}| \sin\theta \, d\theta d\phi \\ &= \frac{1}{|\vec{r}-\vec{r}'|} \int_0^\infty \int_{-\pi}^{\pi} \int_0^{2\pi} e^{i2\pi|\vec{r}-\vec{r}'||\vec{\zeta}|\cos\theta} \frac{1}{4\pi^2} d(|\vec{\zeta}||\vec{r}-\vec{r}'|) \sin\theta \, d\theta d\phi \\ &= \frac{1}{|\vec{r}-\vec{r}'|} \int_0^\infty \int_{-\pi}^{\pi} \int_0^{2\pi} e^{i2\pi t\cos\theta} \frac{1}{4\pi^2} d\,t \sin\theta \, d\theta d\phi \end{aligned} \tag{186}$$

where we have defined $t = |\vec{r}-\vec{r}'||\vec{\zeta}|$, the integral is independent of $|\vec{r}-\vec{r}'|$, and yields a constant. **Evaluate the integral to find the constant** .

19 A bit more challenging is to find the Greens' function corresponding to the differential :

$$(\nabla^2 - m^2)G(\vec{r},\vec{r}') = -\delta^3(\vec{r}-\vec{r}') \tag{187}$$

- As is evident from above discussions, we can use the method of Fourier transforms to solve in-homogeneous Ordinary differential equations with constant coefficients. Let there be a n^{th} order :

$$(a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0)f(x) = g(x) \tag{188}$$

Multiplying both $e^{-i2\pi x\zeta}$ and Fourier transforming :

$$\begin{aligned} &\int_{-\infty}^{\infty} dx e^{-i2\pi x\zeta}(a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0)f(x) = \int_{-\infty}^{\infty} dx e^{-i2\pi x\zeta} g(x) \\ &\Rightarrow (a_n(i2\pi\zeta)^n + a_{n-1}(i2\pi\zeta)^{n-1} + \cdots + a_0)\int_{-\infty}^{\infty} dx e^{-i2\pi x\zeta} f(x) = \int_{-\infty}^{\infty} dx e^{-i2\pi x\zeta} g(x) \end{aligned} \tag{189}$$

Now

$$\tilde{f}(\zeta) = \int_{-\infty}^{\infty} dx e^{-i2\pi x\zeta} f(x) \qquad \tilde{g}(\zeta) = \int_{-\infty}^{\infty} dx e^{-i2\pi x\zeta} g(x) \tag{190}$$

Hence :

$$\begin{aligned} &(a_n(i2\pi\zeta)^n + a_{n-1}(i2\pi\zeta)^{n-1} + \cdots + a_0)\tilde{f}(\zeta) = \tilde{g}(\zeta) \\ &\Rightarrow \tilde{f}(\zeta) = \frac{\tilde{g}(\zeta)}{(a_n(i2\pi\zeta)^n + a_{n-1}(i2\pi\zeta)^{n-1} + \cdots + a_0)} \end{aligned} \tag{191}$$

Thus we find :

$$f(x) = \int_{-\infty}^{\infty} \frac{\tilde{g}(\zeta)}{(a_n(i2\pi\zeta)^n + a_{n-1}(i2\pi\zeta)^{n-1} + \cdots + a_0)} e^{i2\pi x\zeta} d\zeta \tag{192}$$

Which can be evaluated by using Residue theorem .

Special Problems

a) Use the Poisson summation formula to show the following :

$$\phi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} \tag{193}$$

satisfies the following equation :

$$2\phi(x) + 1 = x^{-\frac{1}{2}}(2\phi(\frac{1}{x}) + 1) \tag{194}$$

b) Use the above and the fact that :

$$n^{-s}\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}} = \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1} dx \tag{195}$$

To prove the functional equation for the Riemann Zeta function ($\zeta(s) = \sum_{n=1} n^{-s}$):

$$\zeta(s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}} = \zeta(1-s)\Gamma(\frac{1-s}{2})\pi^{-\frac{1-s}{2}} \tag{196}$$

c) Not from above that if for some $\rho = \sigma + it$, $\zeta(s) = 0$, then from the above functional equation $\zeta(1-\rho)$ must also be zero. This means that both $\rho = \sigma + it$ and $1 - \rho = 1 - \sigma - it$ are zeroes of this function. More over if $\zeta(\rho) = 0$ then so is $\zeta^*(\rho) = \zeta(\rho^*)$, by complex conjugating the above equation. Thus again $\sigma - it$ and $1 - \sigma + it$ must be zeroes of the function $\zeta(\rho)$. These zeroes are called the "non-trivial" zeroes of the zeta function. From the above functional equation, we define $\xi(s) = s(s-1)\zeta(s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}$ and can be written in terms of the "non-trivial" zeroes as follows:

$$s(s-1)\zeta(s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}} = \xi(s) = \prod_{\rho}\left(1 - \frac{s}{\rho}\right) \tag{197}$$

where the product is over all non-trivial zeroes of the zeta function. Hence use the above to find the derivative of the summatory von Mangoldt function in terms of the non-trivial zeroes, which is defined as :

$$\frac{d\psi}{dx} = \sum_{p} \ln p \sum_{n=1}^{\infty} \delta(x - p^n) \tag{198}$$

where the sum is over all primes. For this first prove that :

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p}(1 - p^{-s})^{-1} \tag{199}$$

Now note that the l.h.s converges for $\Re[s] > 1$. Also given is the product form of the Γ function:

$$\Gamma(s) = \frac{1}{s}e^{-\gamma s}\prod_{k=1}^{\infty}(1 + \frac{s}{k})^{-1}e^{\frac{s}{k}} \tag{200}$$

Here γ is the Euler Mascheroni constant.
d) The Riemann Hypothesis states that real part of all the non-trivial zeroes are equal to $\frac{1}{2}$ i.e. $\Re[\rho] = \frac{1}{2}$ for all ρ. If we make the change of variable $s = \frac{1}{1-z}$ then show that the $\Re[s] = \frac{1}{2}$ line is mapped to the unit circle on the comlex plane z , i.e. $|z| = 1$ and the entire right half plane $\Re[s] > \frac{1}{2}$ is mapped to the region inside the unit circle. Hence show that Riemann Hypothesis is true if we have a sequence of numbers $\lambda_n > 0$ for $n = 1, 2, \dots,$ given by :

$$\frac{\lambda_n}{n} = \frac{1}{2\pi i} \oint_c \frac{dz}{z^{n+1}} \ln \xi \left(\frac{1}{1-z} \right) \tag{201}$$

where c is a counter clockwise contour about the origin.